ANALYSIS OF PHASE TRAJECTORIES OF A RANDOM PROCESS

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Abstract

In this article we consider a representation of a random process on phase plane. General notions of a stochastic (random) process and a phase space are introduced and traditional random process analysis methods are being considered. After that a notion of a random process as a phase portrait is given. These phase portraits are described by means of level-crossing characteristics. Finally an application of the specified notion in the field of reliability theory is given.

I. INTRODUCTION

All experimental data collected during an observation of a real physical process could be divided into two categories: deterministic processes and nondeterministic (random, stochastic) processes. [[1]]. Deterministic process could be always explicitly defined by a mathematical relationship. On the other hand it is impossible to predict the next value of a random process. Random processes are described by means of probabilistic concepts and statistical characteristics.

Nondeterministic processes are universal in occurrence; they can be found in many areas of human activity, including science, technics, medicine and especially nature. For example, the following processes are stochastic: height of the wind-induced waves, combustion-chamber pressure in an engine, Brownian motion model of particles and stock market fluctuations, miscellaneous artifacts and noises arising in data acquisition and transmission devices, physiological data such as temperature, pulse, respiration, blood pressure and many others.

II. DEFINITION OF A RANDOM PROCESS

Stochastic process is a collection of random variables $\{\xi(s), s \in S\}$ indexed by parameter s, which belongs to an arbitrary set S [[2] , [3] , [6]]. Most often parameter s is interpreted as time and without loss of generality we may define random process as $\{\xi(t), t \in T\}$, where set T is time. Parameter set of parameter t is considered to be $T = \{t, 0 \le t \le \infty\}$ or $T = \{t, a \le t \le b\}$, in other words it is assumed that T matches either with positive semiaxis $t \in [0; \infty)$ or with a finite interval $t \in [a; b]$ on time axis. Besides that random variables $\xi(t)$ possess values only on the real number line $\xi(t) \in (-\infty; \infty)$.

Function $\xi(t)$, defined on the interval $t \in [a; b]$, would be called a choice function, or a sample distribution function, or realization or trajectory of a random process.

Different random process formulation methods are used depending on the class of current problems. Most often cumulative distribution function (CDF) $F(\xi; t_1)$ and probability density function (PDF) $p(\xi; t_1)$ are used to describe a random value $\xi(t_1)$ at some fixed point in time $t = t_1$.

III. FUNDAMENTAL ANALYSIS METHODS OF A RANDOM PROCESS

As was mentioned above an estimation of random process properties could be made with statistical tools. Let's review basic methods of analysis in order of rising detail level of random process's properties:

[1] Single characteristics.

[2] One-dimensional (univariate) distributions.

[3] Spectral-time dependencies, correlation dependencies.

[4] Multivariate (multi-dimensional) joint distributions.

[5] Level crossing characteristics of a random process.

The expected value (mean) m_ξ, variance $D = \sigma_\xi^2$ and other moment and central moment functions of a random process are all single characteristics of a stochastic process $\xi(t)$.

One-dimensional cumulative distribution function and one-dimensional probability density function give more information about a random process than its moment functions. Besides, one-dimensional probability density function is easily estimated by distribution bar chart method or by level-crossing PDF estimation method [[4]].

Correlation function describes linear dependency between two random values, and finding this dependency is an important task in many applications. It is often role-defining when considering stationary random processes [[3]]. Wiener–Khinchin theorem uniquely relates autocorrelation function of a wide-sense-stationary random process with its power spectral density over a Fourier transform. Hence this theorem allows accomplishing a transition from time characteristics of a random process to its frequency-response characteristics and backwards.

Assembly of all finite dimensional distributions gives a fully complete characteristic of a random process [[2] , [3] , [5]]. Unfortunately multivariate distribution is easy to find only when $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables, in this case probability density function $p_n(\xi_1, \ldots, \xi_n) = = p_{\xi_1}(\xi_1) \cdot p_{\xi_2}(\xi_2) \cdot \ldots \cdot p_{\xi_n}(\xi_n)$, i.e. the task reduces to univariate distributions analysis. In the general case it is very difficult to find a multivariate probability density function p_n or a corresponding cumulative distribution function F_n.

Although finite dimensional distributions give a most complete characteristic of a random process, there is another method for detailed description of actual sample distribution function behavior. This method is concerned with investigation of level crossing characteristics of a process. Stated method will be examined more closely later on in this article.

IV. PHASE TRAJECTORIES AS A METHOD OF DESCRIBING OF COMPLEX DYNAMICAL SYSTEMS

In mathematics and physics a concept of phase space is used to describe multiple states of a system. In phase space a single point describes a state of arbitrarily complex system such as nuclear reactor, space vehicle or a human body, and a movement of this point describes evolution of the system. If we accept that system behavior is characterized by system's state $\xi(t)$ and its rate of change is $\xi'(t) = d\xi(t)/dt$ then the plane $(\xi(t), \xi'(t))$ will be a phase plane (subcase of phase space).

Each point in a phase plane reflects to a single state of the system and is called a phase point, image point or a representative point. The change of the system's state is displayed on the phase plane as a movement of this point. Trace of a moving phase point is called a phase trajectory. Complete collection of different phase trajectories forms a phase portrait. Phase portrait gives a graphical representation of all possible movements in a dynamical system with different initial conditions.

Combined with analytical methods the phase plane method allows obtaining quantitative estimations of solutions of differential equations which describe the dynamical system. For example, the method makes it possible to estimate transition time of the image point from one state to another (i.e. response time), determine oscillation period and amplitude of periodic signal (see 1), etc.

V. DISPLAYING RANDOM PROCESSES ON A PHASE PLANE

The task of studying behavior peculiarities of the phase trajectory $L(\xi, \xi', t)$ of random process $\xi(t)$ on phase plane (ξ, ξ') often arises when dynamics of complex stochastic systems are being examined. In this case the random process is presented in a form of phase trajectory on a phase plane. It should be noted that this description of a random process carries information about both the random process $\xi(t)$ and it's dynamics in the form of derivative $\xi'(t)$. A phase trajectory of random process $\xi(t)$ from 2 is shown on 3.

Although all phase portraits of each sample distribution function of a random process are unique, phase portraits of random processes with the same distribution function are visually similar. Level crossing theory allows for the best way to describe these similarities.

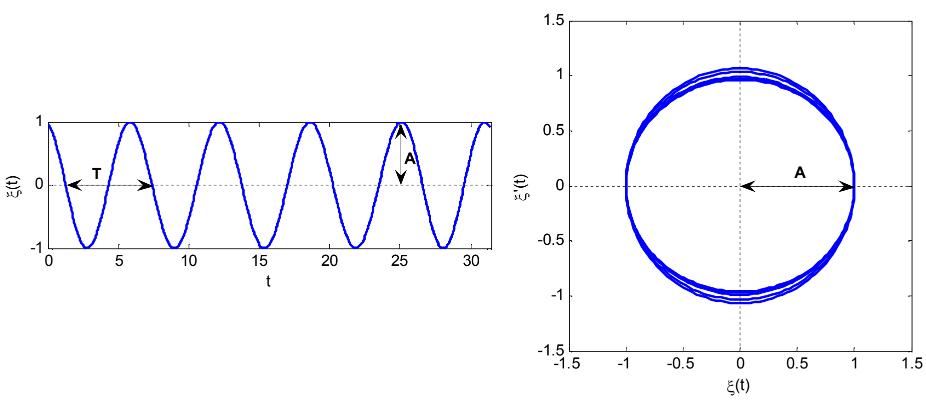

Fig. 1. Simple harmonic motion $\xi(t)$ with random slowly varying phase (left) and its phase portrait (right). A — amplitude of oscillation. Oscillation period T equals to perimeter of a circle on the phase portrait

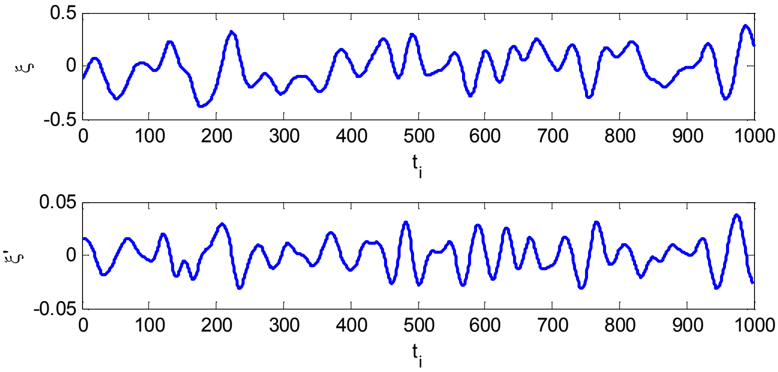

Fig. 2. Sample distribution function of a normally distributed process $\xi(t)$ (top) and it's derivative $\xi'(t)$ (bottom)

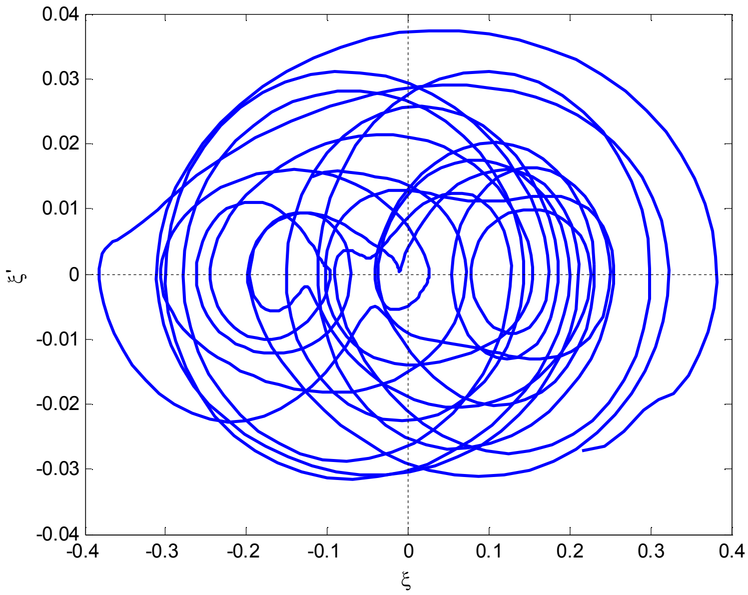

Fig. 3. Phase portrait (phase trajectory) $L(\xi,\xi',t)$ of Gaussian random process $\xi(t)$

VI. EXCURSIONS OF PHASE TRAJECTORY OF A GAUSSIAN PROCESS OUTSIDE A PREDEFINED AREA

Gaussian processes are one of the most frequently encountered classes of random processes. Normal (Gaussian) distribution is used to describe many processes in physics, technics, medicine, biology and other fields. One of the high spots of Gaussian process is that its derivative is also a Gaussian random process. An example of sample distribution function of normally-distributed random process $\xi(t)$ and its derivative $\xi'(t)$ is shown on **2**.

Assume that $L(\xi,\xi',t)$ is a phase trajectory of stationary continuous twice differentiable random process $\xi(t)$. This trajectory represents a two-dimensional process $\xi_1(t)=\{\xi(t),\xi'(t),t\in T\}$. Let's estimate the number of excursions of this two-dimensional process outside a predefined area Ω (see 4). Excursions of process $\xi_1(t)$ outside boundaries AB and CD happen when random process $\xi(t)$ crosses levels $\pm H_1$ on condition that the value of derivative $\xi'(t)$ of random process resides in an interval $[-H_2,H_2]$. Using this line of reasoning it can be obtained that excursions of process $\xi_1(t)$ outside borders BC and AD are due to derivative $\xi'(t)$ of random process $\xi(t)$ crossing levels $\pm H_2$ on condition that $\xi(t)\in[-H_1,H_1]$. A general formula for an average number $N_{\xi_1}(\Omega,T)$ of excursions of phase trajectory $L(\xi,\xi',t)$ outside a predefined area Ω during time interval $[t_0,t_0+T]$ is written as follows:

$$N_{\xi_1}(\Omega,T)=\left[N_{\xi}^{+}(H_1,T)+N_{\xi}^{-}(-H_1,T)\right]P\{\xi'(t)\in[-H_2,H_2]\}+ \\ +\left[N_{\xi'}^{+}(H_2,T)+N_{\xi'}^{-}(-H_2,T)\right]P\{\xi(t)\in[-H_1,H_1]\} \quad (1)$$

where $N^{\pm}(\pm H_i,T)$ is an average number of excursions of one-dimensional processes $\xi(t)$ or $\xi'(t)$ during time interval $[t_0,t_0+T]$ beyond levels $\pm H_1$ or $\pm H_2$ correspondingly.

The following formulas are true for a Gaussian random process [[3]]:

$$P\{\xi(t)\in[-H,H]\}=\Phi\left(\frac{H}{\sigma}\right)-\Phi\left(\frac{-H}{\sigma}\right)=2\Phi\left(\frac{H}{\sigma}\right)-1, \quad (2)$$

$$N^{+}(H,T)=N^{-}(-H,T)=\frac{T}{2\pi}\sqrt{-r''(0)}\exp\left(-\frac{H^2}{2\sigma^2}\right), \quad (3)$$

where $-r_0''=-r_\xi''(0)=-\left.\frac{d^2}{d\tau^2}r_\xi(\tau)\right|_{\tau=0}=\frac{\sigma_{\xi'}^2}{\sigma_\xi^2}$; and $r_\xi(\tau)$ is an autocorrelation function of process $\xi(t)$ normalized by variance. It is connected to standard autocorrelation function $R_\xi(\tau)$ (normalized be mean only) by formula $R_\xi(\tau)=\sigma_\xi{}^2 r_\xi(\tau)$.

Taking into account that derivative of a Gaussian random process is also a Gaussian random process we can easily deduce from formulas (1–3) the following dependency for an average number of excursions of process $\xi_1(t)$ outside area Ω on a unit time interval:

$$\begin{aligned}N_{1\xi_1}(\Omega)&=\frac{1}{T}N_{\xi_1}(\Omega,T)=\\ &=\frac{1}{\pi}\left[\sqrt{-r_\xi''(0)}\exp\left(-\frac{H_1{}^2}{2\sigma_\xi{}^2}\right)\right]\left\{2\Phi\left(\frac{H_2}{\sigma_{\xi'}}\right)-1\right\}+\\ &+\frac{1}{\pi}\left[\sqrt{-r_{\xi'}''(0)}\exp\left(-\frac{H_2{}^2}{2\sigma_{\xi'}{}^2}\right)\right]\left\{2\Phi\left(\frac{H_1}{\sigma_\xi}\right)-1\right\}.\end{aligned} \quad (4)$$

As is shown in [[3]], average value $T_{\xi_1}(\Omega)$ of relative duration of process $\xi_1(t)$ staying in a predefined area Ω would be expressed by formula:

$$T_{\xi_1}(\Omega)=\left[2\Phi\left(\frac{H_1}{\sigma_\xi}\right)-1\right]\left[2\Phi\left(\frac{H_2}{\sigma_{\xi'}}\right)-1\right]. \quad (5)$$

An average duration of an excursion of two-dimensional process $\xi_1(t)$ outside area Ω can be obtained from the following formula:

$$\bar{\tau}_{\xi_1}(\Omega)=\left(1-T_{\xi_1}(\Omega)\right)/N_{1\xi_1}(\Omega). \quad (6)$$

Following formulas can be handy for calculations of excursion characteristics of a random process $\xi(t)$ when processing experimental data:

$$-r_\xi''(0)=\frac{\sigma_{\xi'}{}^2}{\sigma_\xi{}^2},\quad r_{\xi'}''(0)=r_\xi{}^{(4)}(0)=r_0{}^{(4)}=\frac{\sigma_{\xi''}{}^2}{\sigma_{\xi'}{}^2}.$$

VII. EXCURSIONS OF PHASE TRAJECTORY OF A RAYLEIGH PROCESS OUTSIDE A PREDEFINED AREA

Fig. fig. **5** shows an example of sample distribution function of a Rayleigh distributed random process $\xi(t)$ and its derivative $\xi'(t)$. It is worthy of note that one-dimensional PDF of derivative $\xi'(t)$ of Rayleigh distributed process $\xi(t)$ has normal distribution, but in general case for multi-dimensional PDFs of process $\xi'(t)$ this is not true.

Phase portrait of a Rayleigh process $\xi(t)$ is shown on 6. The fact that density of lines forming the phase trajectory $L(\xi,\xi',t)$ of process $\xi(t)$ becomes smaller as the distance to the origin grows, corresponds to excessively long right end of the Rayleigh probability density function plot.

Let's represent phase trajectory $L(\xi,\xi',t)$ of a stationary continuous twice differentiable Rayleigh distributed random process $\xi(t)$ as a two-dimensional process $\xi_2(t)=\{\xi(t),\xi'(t),t\in T\}$ and let's find the number of excursions of this process outside a predefined area Ω (7).

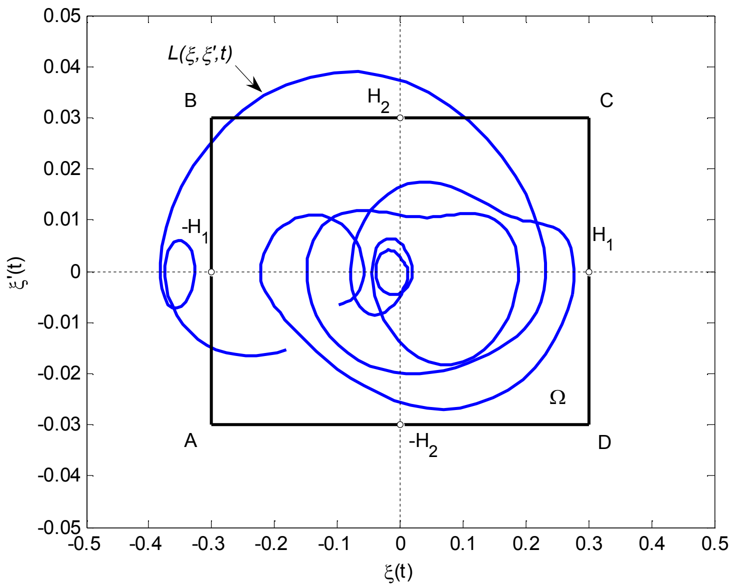

Fig. 4. Trajectory of a two-dimensional vector random process $\xi_1(t)=\{\xi(t),\xi'(t)\}$ on a phase plane (ξ,ξ').

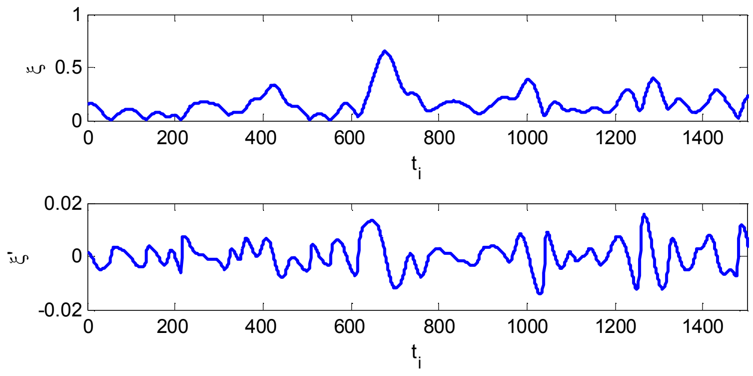

Fig. 5. Sample distribution function of a random process with Rayleigh distribution (top) and its derivative (bottom)

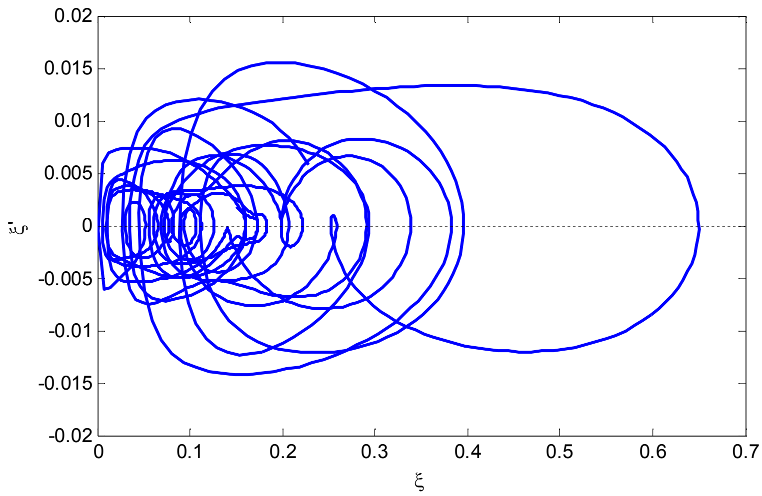

Fig. 6. Phase portrait (phase trajectory) $L(\xi, \xi', t)$ of a Rayleigh distributed random process

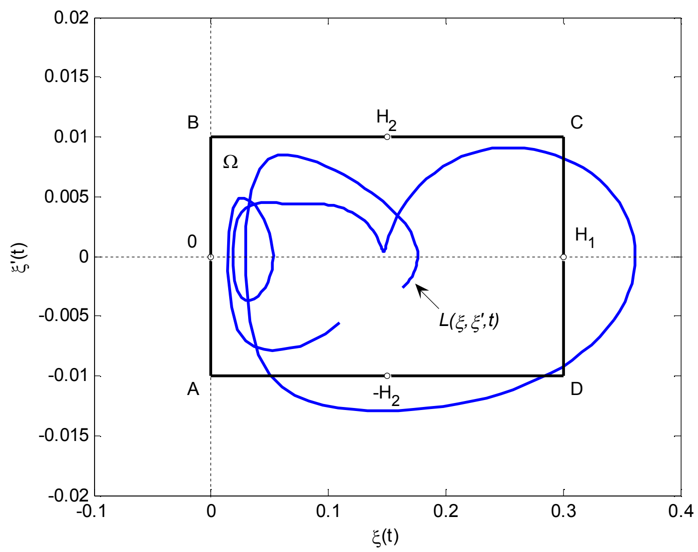

Fig. 7. Excursions of Rayleigh process outside predefined area Ω .

For a Rayleigh random process the following two formulas take place:

$$P\{\xi(t) \in [0, H]\} = 1 - \exp\left(-H^2 / 2\sigma^2\right), \qquad (7)$$

$$N^+(H,T) = T\left(\frac{-\rho''(0)}{2\pi}\right)^{1/2}\left(2 - \frac{\pi}{2}\right)^{1/2}\frac{H}{\sigma}\exp\left(-\frac{H^2}{2\sigma^2}\right), \qquad (8)$$

where $\rho(\tau)$ is a normalized by variance autocorrelation function of process $\xi(t)$, it is connected to standard autocorrelation function $R_\xi(\tau)$ by formula $R_\xi(\tau) = \sigma_\xi^2 \rho(\tau)$.

Using the line of reasoning stated in the previous part of this article, based on formulas (1–3, 7 and 8) for a Rayleigh process we will receive:

$$N_{1\xi_2}(\Omega)=\frac{1}{T}N_{\xi_2}(\Omega,T)=\left(\frac{-\rho''(0)}{2\pi}\right)^{1/2}\left(2-\frac{\pi}{2}\right)^{1/2}\times$$
$$\times\frac{H_1}{\sigma}\exp\left(-\frac{H_1^2}{2\sigma^2}\right)\left[2\Phi\left(\frac{H_2}{\sigma_{\xi'}}\right)-1\right]+ \quad (9)$$
$$+\frac{1}{\pi}\left[\sqrt{-r''_{\xi'}(0)}\exp\left(-\frac{H_2^2}{2\sigma_{\xi'}^2}\right)\right]\left\{1-\exp\left(\frac{-H_1^2}{2\sigma^2}\right)\right\},$$

where σ is a parameter of Rayleigh distribution (its mode), which is connected with variance σ_ξ^2 of Rayleigh distributed process $\xi(t)$ by formula $\sigma_\xi^2=(2-\pi/2)\sigma^2$.

For an average value $T_{\xi_2}(\Omega)$ of relative duration of process $\xi_2(t)$ staying inside area Ω and for an average duration of excursions of two-dimensional process $\xi_2(t)$ we deduce [[3]]:

$$T_{\xi_2}(\Omega)=\left[1-\exp\left(\frac{-H_1^2}{2\sigma^2}\right)\right]\left[2\Phi\left(\frac{H_2}{\sigma_{\xi'}}\right)-1\right], \quad (10)$$
$$\bar{\tau}_{\xi_2}(\Omega)=\left(1-T_{\xi_2}(\Omega)\right)/N_{1\xi_2}(\Omega). \quad (11)$$

VIII. PRACTICAL EXAMPLE

Let us deal with the following practical example. Assume that we have a dynamic system and its states are described by two-dimensional random function $\eta(t)=\{\eta_1(t),\eta_2(t)\}$. The system is considered functioning at the point of time $t=t_1$, if $\eta(t_1)\in\Omega$, where Ω is a set of all functional (correct) states of the system. We will assume that borders of the set Ω can be defined as a rectangle on a plane (as shown on 4 and 7).

Now let's formulate a classical problem in reliability theory — estimate the number of failures of our system during a specified time interval.

To solve this problem we will assume that variables $\eta_1(t)$ and $\eta_2(t)$ are both described by normal distribution law with mean $m_{\eta_1}=m_{\eta_2}=0$ and variances $\sigma_{\eta_1}^2=\sigma_1^2$ and $\sigma_{\eta_2}^2=\sigma_2^2$ respectively, and their autocorrelation functions are in the form $R_{\eta_i}(\tau)=\sigma_i^2 r(\tau)=\sigma_i^2\exp\left(-\alpha\tau^2\right), i=\overline{1,2}$.

Now we can apply methods, discussed in section VI. Average number of excursions of process $\eta(t)$ outside acceptance area Ω during interval of analysis $[t_0,t_0+T]$ will be defined as follows:

$$N_\eta(\Omega,T)=\frac{T}{\pi}\left[\sqrt{2\alpha}\exp\left(-\frac{H_1^2}{2\sigma_1^2}\right)\right]\left\{2\Phi\left(\frac{H_2}{\sigma_2}\right)-1\right\}+$$
$$+\frac{T}{\pi}\left[\sqrt{2\alpha}\exp\left(-\frac{H_2^2}{2\sigma_2^2}\right)\right]\left\{2\Phi\left(\frac{H_1}{\sigma_1}\right)-1\right\}. \quad (12)$$

Characteristic $N_\eta(\Omega,T)$ in the scope of current task allows to calculate the number of system failures in a definite time T, and value $\frac{1}{T}N_\eta(\Omega,T)=N_{1\eta}(\Omega)$ is the failure rate (number of failures during a unit time). Characteristic $\bar{\tau}_\eta(\Omega)$ is the average recovery time and $T_\eta(\Omega)$ — mean time between failures.

IX. CONCLUSION

The examined method of random process analysis on a phase plane makes it possible to graphically represent a random process as well as its dynamics, thus reducing the complexity of analysis of complex stochastic systems. Use of level-crossing theory for analysis of phase portraits allowed us to describe detailed structure of a random process. Proposed implementation of all computations is easily achieved both on software and hardware level as the main computational component is a counter of level-crossings.

Excursion characteristics of random process outside a predefined area may be used in control systems, diagnostic systems, decision-making systems, for classification and other tasks.

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